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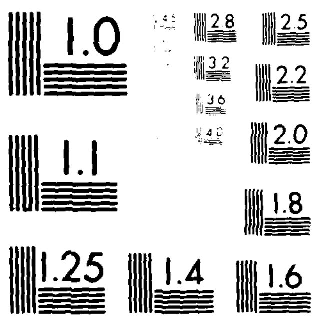

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SYSTEMS NOTE 62

CONVERGENCE BEHAVIOUR OF SOME ITERATION PROCEDURES FOR EXTERIOR POINT METHOD OF CENTRES ALGORITHMS

by

RONALD B. ZMOOD

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ERRATA

In general for equations with limit $\Big|_{t=\bar{v}}$ should read $\Big|_{t=\bar{v}}$

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by

(10) RONALD B. ZMOOD

SUMMARY

The convergence rate for a number of iterative procedures for the method of centres, was studied in connection with the investigation of methods for extending the applicability of flight directors. By the use of the Kuhn-Tucker conditions and the duality properties for convex programming problems, it was shown that the augmented cost function, arising in this method, has a second order zero at the optimum point. From this flows the results: that the Staha and Morrison iteration procedures are linearly convergent; the tangent iteration procedure is quadratically convergent; and two interpolation polynomial iteration procedures proposed by the author to overcome the deficiencies of the tangent method away from the optimum point are super-linearly convergent and are thus worthy of further investigation.

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16. ABSTRACT

The convergence rate for a number of iterative procedures for the method of centres, was studied in connection with the investigation of methods for extending the applicability of flight directors. By the use of the Kuhn-Tucker conditions and the duality properties for convex programming problems, it was shown that the augmented cost function, arising in this method, has a second order zero at the optimum point. From this flows the results: that the Staha and Morrison iteration procedures are linearly convergent; the tangent iteration procedure is quadratically convergent; and two interpolation polynomial iteration procedures proposed by the author to overcome the deficiencies of the tangent method away from the optimum point are super-linearly convergent and are thus worthy of further investigation.

CONTENTS

1. INTRODUCTION

The range of applicability of flight directors to situations involving large manoeuvres and disturbances is capable of being extended by the incorporation of predictive methods in digital avionic and display systems. This involves the determination of optimal controls in the presence of physical and operational constraints, and alternative numerical algorithms for this purpose are being investigated. In this note the speed of convergence for the method of centres optimization algorithm, which can handle the above types of constraints, is examined to determine its suitability for rapid computations.

The method of centres algorithm is a well established technique for finding the minimum of a function of n variables subject to inequality constraints. Lootsma[1] distinguishes two classes of such algorithms, namely: interior and exterior points methods. For the former methods extensive theoretical and computational studies, of their convergence properties and rates of convergence, have been made by Huard,[2] Faure and Huard,[3] Lootsma,[1,4] and Pironneau and Polak.[5] In particular Lootsma,[1] and Pironneau and Polak[5] have shown that the algorithms proposed by Huard are linearly convergent.

Exterior point methods, the subject of this report, were first proposed by Kowalik,[8] and have been studied both theoretically and computationally by Morrison,[6] Kowalik *et al.*[7] and Lootsma.[1,4] Numerical experience with a number of these methods has been variable, with Lootsma reporting disappointing results for the algorithm considered in Reference 4, while Staha and Himmelblau[9] reported very favourably on their application of Newton's method to the exterior point method of centres. Lootsma[10] studied the theoretical rates of convergence for a number of exterior point algorithms including the one proposed by Morrison,[6] the tangent algorithm discussed by Kowalik *et al.*[7] and the Newton method discussed by Staha.[9] He concluded that the Morrison and Staha methods are linearly convergent while the tangent method is super-linearly convergent. Numerical experience, however, has shown that Staha's method usually converges more rapidly than the tangent method, which tends to overshoot the optimum solution. This apparent paradox between the Staha and tangent methods is even more compounded when it is noted that their iteration formulae only differ by a factor of one half.

In this report we investigate the convergence behaviour of the Morrison, tangent, and Staha algorithms, as well as two interpolation polynomial algorithms proposed by the author. The approach used, which differs from that of Lootsma, not only confirms his results for the Morrison and Staha methods, but also gives further insight into the reason why the Staha algorithm is only linearly convergent, yet the tangent algorithm is super-linearly convergent. In fact we are able to demonstrate that the tangent algorithm is quadratically convergent as well as obtaining an explicit formula for its asymptotic error constant. In addition we are able to show that one of the proposed algorithms is almost quadratically convergent. This appears to confirm the limited computational experience obtained to date which shows that its convergence rate for test problems compares very favourably with the tangent method.

2. PROBLEM DEFINITION AND PRELIMINARY RESULTS

In this report we shall consider exterior point method of centres algorithms for solving the following constrained minimization problem:

PROBLEM MP. Find a point $\bar{x} \in C$ where $C = \{\bar{x} \in R^n \mid f_i(x) \leqslant 0,\ i = 1, \ldots, m\}$ such that

$$f_0(\bar{x}) = \min_{x \in C} f_0(x). \tag{2.1}$$

We shall assume that the functions $f_i(\cdot)$, $i = 0, 1, \ldots, m$ mapping R^n into R^1 are twice continuously differentiable. The gradient and hessian of a function f_i at $\bar{x} \in R^n$ will be denoted by $f_{ix}(\bar{x})$ and $f_{ixx}(\bar{x})$ respectively. The gradient will be considered to be a row vector and will be written as

$$f_{ix}(\bar{x}) = \left(\frac{\partial f_i(\bar{x})}{\partial x_1}, \ldots, \frac{\partial f_i(\bar{x})}{\partial x_n}\right), \tag{2.2}$$

while the kl'th element of the hessian will be written as

$$[f_{ixx}(\bar{x})]_{kl} = \frac{\partial^2 f_i(\bar{x})}{\partial x_k \, \partial x_l}, \quad k, l = 1, \ldots, n. \tag{2.3}$$

The set C, defined above, will be referred to as the *constraint set* and it will be assumed that it is compact with a non-empty interior. Any point $x \in C$ will be referred to as an *admissible solution* of problem MP. From the compactness assumption on C and the fact that f_0 is continuous it follows that an $\bar{x} \in C$ exists satisfying (2.1).

Problem MP with the additional assumption that the functions $f_i(\cdot)$, $i = 0, 1, \ldots, m$ are convex on R^n, will be termed a convex programming problem.

In considering the method of centres we introduce the augmented cost function $Q_t(x)$, dependant on the parameter $t \in R^1$ and the variable $x \in R^n$, which is defined in terms of the functions $f_i(x)$, $i = 0, 1, \ldots, m$ given in problem MP above. This function is defined as

$$Q_t(x) = [\max(0, f_0(x) - t)]^2 + \sum_{i=1}^{m} [\max(0, f_i(x))]^2. \tag{2.4}$$

For any t we denote the point $x \in R^n$ minimizing $Q_t(x)$ as $x(t)$. The significance of this definition of $Q_t(x)$ follows from Lootsma's[1] result that, under appropriate assumptions, if a sequence of parameters t_i, $i = 1, 2, \ldots$ converges to $\bar{v} = f(\bar{x})$ then the corresponding sequence of points $x(t_i) \in R^n$, $i = 1, 2, \ldots$ converges to $\bar{x}$. In the sequel we will study the rate of convergence of several algorithms for adjusting the parameter t so that the sequence of points $\{x(t_i)\}$ generated by it approaches $\bar{x}$ from the exterior of the constraint set C.

Before proceeding we state some basic results which will be used in later sections.

THEOREM 2.1. Suppose $f_i(\cdot) \in C^1(R^n; R^1)$, $i = 0, 1, \ldots, m$ and that problem MP is a convex programming problem. In addition assume that the interior of the constraint set C is non-empty. Then an admissible solution $\bar{x}$ is a minimum solution of problem MP if and only if there exists a $\bar{u} \in R^m$ such that

$$f_i(\bar{x}) \leqslant 0, \tag{2.5}$$

$$\bar{u}_i \geqslant 0, \tag{2.6}$$

$$\bar{u}_i f_i(\bar{x}) = 0, \tag{2.7}$$

for $i = 1, \ldots, m$, and

$$f_{0x}(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i f_{ix}(\bar{x}) = 0. \tag{2.8}$$

Proof. See Lootsma (Ref. 1, p. 25).

DEFINITION 2.1. A point $(\bar{x}, \bar{u}) \in R^n \times R^m$ is termed a Kuhn-Tucker point of problem MP *if it satisfies the conditions (2.5) to (2.8).*

We introduce the following notation: Let $I(\bar{x}) = \{i \mid f_i(\bar{x}) = 0, i = 1, \ldots, m\}$ denote the set of active constraints. The derivative of the left hand side of (2.8) will be denoted by

$$D^2(\bar{x}, \bar{u}) = f_{0xx}(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i f_{ixx}(\bar{x}). \tag{2.9}$$

Following Lootsma we give the following definition:

DEFINITION 2.2. A Kuhn-Tucker point $(\bar{x}, \bar{u})$ of problem MP satisfies the *Jacobian Uniqueness Condition* if:

(*a*) $\bar{u}_i > 0$ for $i \in I(\bar{x})$,

(*b*) $f_{ix}(\bar{x})$, $i \in I(\bar{x})$ are linearly independent, and

(*c*) for every non-zero $y \in R^n$, $f_{ix}(\bar{x})y = 0$, $i \in I(\bar{x})$ implies $y^T D^2(\bar{x}, \bar{u})y > 0$.

THEOREM 2.2. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$ and there exists a Kuhn-Tucker point $(\bar{x}, \bar{u})$ satisfying the Jacobian uniqueness condition then the point $\bar{x}$ is an isolated local minimum of problem MP and the vector $\bar{u}$ is uniquely determined.

Proof. See Lootsma (Reference 1, p. 17).

We now introduce the dual problem to problem MP.

PROBLEM DP. Suppose $f_i(\cdot) \in C^1(R^n; R^1)$, $i = 0, 1, \ldots, m$. Find $\hat{x}$ and $\hat{u} \in R^m$ such that

$$f_0(\hat{x}) + \sum_{i=1}^{m} \hat{u}_i f_i(\hat{x}) = \max_{(x,u) \in Y} \left(f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \right) \tag{2.10}$$

where

$$(\hat{x}, \hat{u}) \in Y = \left\{ (x, u) \mid x \in R^n, u \in R^m, f_{0x}(x) + \sum_{i=1}^{m} u_i f_{ix}(x) = 0, u_i \geqslant 0, i = 1, \ldots, m \right\}.$$

We define

$$\psi(x, u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \tag{2.11}$$

Wolfe[11] has proved the following result:

THEOREM 2.3. Suppose $f_i(\cdot) \in C^1(R^n; R^1)$, $i = 0, 1, \ldots, m$ and that problem MP is a convex programming problem. If $\bar{x}$ is a minimum of problem MP and satisfies a suitable constraint qualification condition then there exists a $\bar{u} \in R^m$ such that $(\bar{x}, \bar{u})$ satisfies the dual problem and also

$$f_0(\bar{x}) = \psi(\bar{x}, \bar{u}). \tag{2.12}$$

Proof. See Mangasarin.[16]

The reader is referred to Mangasarin (Ref. 16, p. 105) for a discussion of suitable constraint qualification conditions.

3. PROPERTIES OF AUGMENTED COST FUNCTION

In this section we examine some of the properties of the augmented cost function $Q_t(x)$.

Under the assumptions given in Section 2, and including the assumption that problem MP is a convex programming problem, Lootsma[1] has shown that $Q_t(x)$ has the following properties:

(*a*) for every t there exists an $x(t) \in R^n$ minimizing $Q_t(x)$,

(*b*) if $t < \bar{v}$ then $t < f_0(x(t)) \leqslant \bar{v}$ and $Q_t(x(t)) > 0$, and

(*c*) if $t \geqslant \bar{v}$ then $Q_t(x(t)) = 0$.

In addition we need the following results shown by Lootsma.

THEOREM 3.1. Suppose $f_i(\cdot) \in C^1(R^n; R^1)$, $i = 0, 1, \ldots, m$ and that problem MP is a convex programming problem. Then for every $t \leqslant \bar{v}$ an admissible solution of problem DP is given by $(x(t), u(t))$ where $x(t)$ minimizes $Q_t(x)$ over R^n and $u(t)$ is given by

$$u_i(t) = \frac{\max\,[0, f_i(x(t))]}{f_0(x(t)) - t}, \quad i = 1, \ldots, m. \tag{3.1}$$

Proof. This may be shown by direct substitution.

THEOREM 3.2. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$, problem MP is a convex programming problem and there exists a Kuhn-Tucker point $(\bar{x}, \bar{u})$ which satisfies the Jacobian uniqueness condition, then

$$\lim_{\substack{t \to \bar{v} \\ t < \bar{v}}} (x(t), u(t)) = (\bar{x}, \bar{u}). \tag{3.2}$$

Proof. This may be shown by making the obvious modifications to the proof given in Lootsma (Ref. 1, p. 38).

LEMMA 3.1. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$, problem MP is a convex programming problem and these exists a Kuhn-Tucker point $\bar{y} = (\bar{x}, \bar{u}) \in R^n \times R^m$ which satisfies the Jacobian uniqueness condition, then for $t \leqslant \bar{v}$

$$y(t) = \bar{y} - F_y(\bar{y}; \bar{t})^{-1} F_t(\bar{y}, \bar{t})(t - \bar{v}) + O[(t - \bar{v})^2], \tag{3.3}$$

where

$$
F(y;t) = \left[\begin{array}{c}
f_{0x}^T(x) + \sum_{i=1}^{m} u_i f_{ix}^T \\
\hline
u_1 f_1(x) - \dfrac{\max(0, f_1(x))}{f_0(x) - t} f_1(x) \\
\vdots \\
u_m f_m(x) - \dfrac{\max(0, f_m(x))}{f_0(x) - t} f_m(x)
\end{array}\right], \tag{3.4}
$$

and $y(t) = (x(t), u(t))$.

Proof. To investigate the behaviour of $(x(t), u(t))$ for t in a neighbourhood of $\bar{v}$, we note from the definition of $x(t)$ and (2.4) that for $t \leqslant \bar{v}$

$$
f_{0x}(x(t))[\max(0, f_0(x(t)) - t)] + \sum_{i=1}^{m} f_{ix}(x(t))[\max(0, f_i(x(t)))] = 0. \tag{3.5}
$$

From theorem 3.1 this is equivalent to

$$
f_{0x}(x(t)) + \sum_{i=1}^{m} u_i(t) f_{ix}(x(t)) = 0, \tag{3.6}
$$

$$
u_i(t) f_i(x(t)) - \frac{\max[0, f_i(x(t))]}{f_0(x(t)) - t} f_i(x(t)) = 0,\ i = 1, \ldots, m, \tag{3.7}
$$

for $t \leqslant \bar{v}$. Equations (3.6) and (3.7) implicitly define $(x(t), u(t))$ in terms of the parameter t.

Without loss of generality assume that the set of active constraints $I(\bar{x}) = \{1, \ldots, \alpha\}$. From theorem 2.1 and the Jacobian uniqueness conditions it follows that

$$
f_i(\bar{x}) = 0 \text{ and } \bar{u}_i > 0 \quad \text{for } i = 1, \ldots, \alpha, \tag{3.8}
$$

and

$$
f_i(\bar{x}) < 0 \text{ and } \bar{u}_i = 0 \quad \text{for } i = \alpha + 1, \ldots, m. \tag{3.9}
$$

The continuity of the functions $f_i(x)$, $i = 1, \ldots, m$, and the results of theorems 3.1 and 3.2 lead to the conclusion that there exists an $\varepsilon > 0$ so that for $t \in [\bar{v} - \varepsilon, \bar{v})$,

$$
f_i(x(t)) > 0 \text{ and } u_i(t) > 0 \quad \text{for } i = 1, \ldots, \alpha, \tag{3.10}
$$

and

$$
f_i(x(t)) < 0 \text{ and } u_i(t) = 0 \quad \text{for } i = \alpha + 1, \ldots, m. \tag{3.11}
$$

From (3.4) it can be seen that $F(\bar{x}, \bar{u}; \bar{v}) = 0$, and also that the matrix partial derivative of F with respect to y is

$$
F_y(y;t) = \left[\begin{array}{c|ccc|ccc}
D^2(x,u) & f_{1x}^T & \cdots & f_{\alpha x}^T & f_{\alpha+1 x}^T & \cdots & f_{mx}^T \\
\hline
u_1 f_{1x}(x) - \dfrac{2(f_0 - t) f_1 f_{1x} - f_1 f_{0x}}{(f_0 - t)^2} & f_1 & & 0 & & & \\
\vdots & & \ddots & & & 0 & \\
u_\alpha f_{\alpha x}(x) - \dfrac{2(f_0 - t) f_\alpha f_{\alpha x} - f_\alpha f_{0x}}{(f_0 - t)^2} & 0 & & f_\alpha & & & \\
\hline
 & & & & f_{\alpha+1} & & \\
0 & & 0 & & & \ddots & \\
 & & & & & & f_m
\end{array}\right]. \tag{3.12}
$$

It follows from the argument given by Lootsma (Ref. 1; p. 17) that the matrix $F_y(\bar{y}; \bar{v})$ is non-singular. By the implicit function theorem[12] there exists a neighbourhood, N, of $\bar{v}$ and a unique function $y(t) \in C^2(N; R^{n+m})$ such that $y(\bar{v}) = \bar{y}$, and $F(y(t); t) = 0$ for $t \in N$.

From Taylor's theorem and noting that

$$F_y(\bar{y}; \bar{v}) \frac{dy}{dt}\bigg|_{t=\bar{v}} + F_t(\bar{y}; \bar{v}) = 0, \tag{3.13}$$

we have

$$y(t) = \bar{y} + F_y(\bar{y}; \bar{v})^{-1} F_t(\bar{y}; \bar{v})(t - \bar{v}) + O[(t - \bar{v})^2]. \tag{3.14}$$

LEMMA 3.2. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$, problem MP is a convex programming problem and there exists a Kuhn-Tucker point $(\bar{x}, \bar{u})$ which satisfies the Jacobian uniqueness condition, then

$$\frac{d}{dt} f_0(x(t))\bigg|_{t=\bar{v}-} = \frac{\bar{\beta}}{1 + \bar{\beta}},$$

and

$$\frac{d^2}{dt^2} f_0(x(t))\bigg|_{t=\bar{v}-} = \frac{2\bar{u}^T E_2 F_y^{-1}(\bar{y}; \bar{v}) F_t(\bar{y}; \bar{v})}{(1 + \bar{\beta})^2} +$$

$$+ \frac{1}{1 + \bar{\beta}} (E_1 F_y^{-1}(\bar{y}; \bar{v}) F_t(\bar{y}; \bar{v}))^T D^2(\bar{x}, \bar{u})(E_1 F_y^{-1}(\bar{y}; \bar{v}) F_t(\bar{y}; \bar{v})),$$

where

$\bar{\beta} = \sum_{i \in I(\bar{x})} \bar{u}_i^2$, and the $n \times (m+n)$ matrix E_1 and the $m \times (m+n)$ matrix E_2 are defined as

$$E_1 = [I \,\vdots\, 0] \text{ and } E_2 = [0 \,\vdots\, I],$$

respectively.

Proof. From the assumptions given above it follows that (3.10) and (3.11) hold for $i \in I(\bar{x})$ and $i \notin I(\bar{x})$, respectively.

Considering the expression

$$f_0(x(t)) + \sum_{i \in I(\bar{x})} \bar{u}_i f_i(x(t)), \tag{3.15}$$

and expanding it into a Taylor series we have

$$f_0(x(t)) + \sum_{i \in I(\bar{x})} \bar{u}_i f_i(x(t)) = f_0(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{u}_i f_i(\bar{x}) + \left[f_{0x}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{u}_i f_{ix}(\bar{x}) \right] (x(t) - \bar{x}) +$$

$$+ \tfrac{1}{2}(x(t) - \bar{x})^T D^2(\bar{x}, \bar{u})(x(t) - \bar{x}) + \gamma(\bar{x}, x(t)) ||x(t) - \bar{x}||^2, \tag{3.16}$$

where $\gamma(\bar{x}, x(t)) \to 0$ as $x(t) \to \bar{x}$, and $||\cdot||$ denotes the Euclidean norm on R^n. Consequently from (3.10), (3.11), equation (3.16) and theorems 2.1 and 3.1 we have

$$f_0(x(t)) + (f_0(x(t)) - t) \sum_{i \in I(\bar{x})} \bar{u}_i u_i(t) =$$

$$= f_0(\bar{x}) + \tfrac{1}{2}(x(t) - \bar{x}) D^2(\bar{x}, \bar{u})(x(t) - \bar{x}) + \gamma(\bar{x}, x(t)) \, ||x(t) - \bar{x}||^2. \tag{3.17}$$

Defining $\beta(t) = \sum_{i \in I(\bar{x})} \bar{u}_i u_i(t)$, (3.17) becomes

$$f_0(x(t)) = f_0(\bar{x}) + \frac{\beta(t)}{1 + \beta(t)}(t - \bar{v}) + \frac{1}{2(1 + \beta(t))}(x(t) - \bar{x}) D^2(\bar{x}, \bar{u})(x(t) - \bar{x}) +$$

$$+ \frac{\gamma(\bar{x}, x(t))}{1 + \beta(t)} ||x(t) - \bar{x}||^2. \tag{3.18}$$

Let us define the $n \times (n+m)$ matrix E_1 and the $m \times (n+m)$ matrix E_2 as

$$E_1 = [I \,\vdots\, 0] \text{ and } E_2 = [0 \,\vdots\, I],$$

respectively. From lemma 3.1 we have

$$x(t) - \bar{x} = -E_1 F_y^{-1}(\bar{y}; \bar{v}) \, F_t(\bar{y}; \bar{v})(t - \bar{v}) + O[(t - \bar{v})^2], \tag{3.19}$$

so that

$$f_0(x(t)) = f_0(\bar{x}) + \frac{\beta(t)}{1+\beta(t)}(t - \bar{v}) +$$

$$+ \frac{1}{2(1+\beta(t))}(E_1 F_y^{-1}(\bar{y};\bar{v})\, F_t(\bar{y};\bar{v}))^T D^2(\bar{x},\bar{u})(E_1 F_y^{-1}(\bar{y};\bar{v})\, F_t(\bar{y};\bar{v}))(t-\bar{v})^2 + O[(t-\bar{v})^3]. \tag{3.20}$$

Differentiating (3.20) with respect to t, we find

$$\frac{d}{dt} f_0(x(t))\bigg|_{t=\bar{v}-} = \frac{\bar{\beta}}{1+\bar{\beta}}, \text{ and} \tag{3.21}$$

$$\frac{d^2}{dt^2} f_0(x(t))\bigg|_{t=\bar{v}-} = \frac{2\bar{u}^T E_2 F_y^{-1}(\bar{y};\bar{v})\, F_t(\bar{y};\bar{v})}{(1+\bar{\beta})^2} +$$

$$+ \frac{1}{(1+\bar{\beta})}(E_1 F_y^{-1}(\bar{y};\bar{v})\, F_t(\bar{y};\bar{v}))^T D^2(\bar{x},\bar{u})(E_1 F_y^{-1}(\bar{y};\bar{v})\, F_t(\bar{y};\bar{v})). \tag{3.22}$$

THEOREM 3.3. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$, problem MP is a convex programming problem, and there exists a Kuhn-Tucker point $(\bar{x}, \bar{u})$ which satisfies the Jacobian uniqueness conditions. Then $Q_t(x(t))$ has a second order zero at $t = \bar{v}$.

Proof. Considering $Q_t(x(t))$ we find

$$\frac{d}{dt} Q_t(x(t)) = 2 \max(0, f_0(x(t)) - t)\left(\frac{dx}{dt} f_{0x}(x(t)) - 1\right) + 2 \sum_{i=1}^{m} \max[0, f_i(x(t))] \frac{dx}{dt} f_{ix}(x(t)). \tag{3.23}$$

Since $x(t)$ is a stationary point of $Q_t(x(t))$ we have

$$\max(0, f_0(x(t)) - t) f_{0x}(x(t)) + \sum_{i=1}^{m} \max[0, f_i(x(t))] f_{ix}(x(t)) = 0, \tag{3.24}$$

so that

$$\frac{d}{dt} Q_t(x(t)) = -2 \max(0, f_0(x(t)) - t). \tag{3.25}$$

Now from theorem 3.2 as $t \to \bar{v}$ $(t < \bar{v})$ then $x(t)$ converges to $\bar{x}$, so that

$$\frac{d}{dt} Q_t(x(t))\bigg|_{t=\bar{v}-} = 0.$$

From (3.25) we have for $t < \bar{v}$ that

$$\frac{d}{dt} Q_t(x(t)) = -2(f_0(x(t)) - t). \tag{3.26}$$

Thus

$$\frac{d^2}{dt^2} Q_t(x(t))\bigg|_{t=\bar{v}-} = -2\left(\frac{d}{dt} f_0(x(t))\bigg|_{t=\bar{v}-} - 1\right), \tag{3.27}$$

$$= \frac{2}{1+\bar{\beta}}.$$

Since $\bar{\beta} \geqslant 0$ it follows that the right hand side of (3.27) is always non-zero, and as a consequence $Q_t(x(t))$ has a second order zero at $t = \bar{v}$.

4. LOCAL CONVERGENCE BEHAVIOUR OF ITERATION ALGORITHMS

In this section we examine the local rate of convergence for a number of iterative algorithms which adjust the parameter t of function $Q_t(x)$ so that the sequence t_i, $i = 1, 2, \ldots$ converges to $\bar{v}$, and the corresponding sequence $x(t_i)$ converges to $\bar{x}$, the optimum solution of problem MP.

Because the function $Q_t(x(t))$ has the property,

$$Q_t(x(t)) \begin{cases} = 0 & \text{for all } t \geqslant \bar{v}, \\ > 0 & \text{for all } t < \bar{v}, \end{cases} \tag{4.1}$$

it has a non-unique minimum. Consequently procedures for generating the sequence $t_i, i = 1, 2, \ldots$ must begin with $t_1 < \bar{r}$. It is possible to demonstrate[10] that, under appropriate assumptions, the sequences $t_i, i = 1, 2, \ldots$ generated by the procedures discussed below are monotonic increasing, satisfy the inequalities

$$t_i < t_{i+1} \leqslant \bar{r}, \tag{4.2}$$

and converge to $\bar{r}$.

As a measure of the rate of convergence we use the definition of the order of an iteration algorithm,

$$t_{i+1} = \psi(t_i), \; i = 1, 2, \ldots, \tag{4.3}$$

due to Traub,[13] where $\psi(t)$ will be termed the iteration function.

DEFINITION 4.1. If $t_i, i = 1, 2, \ldots$ is a sequence converging to $\bar{r}$, generated by (4.3), and there exists a $p \geqslant 0$ and a non zero constant C such that

$$\frac{|t_{i+1} - \bar{r}|}{|t_i - \bar{r}|^p} \to C \tag{4.4}$$

then p and C are termed the order of the iteration function and the asymptotic error constant respectively.

The following theorem proved in Traub[13] will be used in the sequel.

THEOREM 4.1. If there exists a neighbourhood, N, of $\bar{r}$ such that $\psi(\cdot) \in C^p(N; R^1)$ then ψ is of order p if and only if

$$\psi(\bar{r}) = \bar{r}, \; \psi^{(j)}(\bar{r}) = 0, \; j = 1, \ldots, p-1 \text{ and } \psi^{(p)}(\bar{r}) \neq 0.$$

In addition the asymptotic error constant is given by

$$\frac{t_{i+1} - \bar{r}}{|t_i - \bar{r}|^p} \to \frac{\psi^{(p)}(\bar{r})}{p!}. \tag{4.5}$$

4.1 The Morrison Algorithm

In reference 6, Morrison proposed the iteration function

$$\psi(t) = t + \lambda Q_t(x(t)) \tag{4.6}$$

for adjusting the parameter t in the method of centres. It will be observed that $\psi(\bar{r}) = \bar{r}$. Also from (4.6), theorem 3.1 and lemma 3.2, we find that

$$\psi^{(1)}(\bar{r}) = 1 - \frac{1}{\lambda(1 + \bar{\beta})}, \tag{4.7}$$

where $\bar{\beta}$ is defined in lemma 3.2. Providing the set of active constraints $I(\bar{x})$ is non-empty it follows from theorem 4.1 that the Morrison algorithm has order $p = 1$, and so is linearly convergent. Furthermore the asymptotic error constant is given by (4.7). These results concur with those derived by Lootsma[10] by different means.

In the case where $I(\bar{x}) = \phi$, that is where problem MP has an unconstrained minimum in C, the method of centres using the Morrison algorithm converges in one step.

4.2 The Staha Algorithm

Staha[14] considered the application of the Newton-Raphson root finding procedure to the method of centres. In this case the iteration function is

$$\psi(t) = t - \frac{Q_t(x(t))}{\frac{d}{dt} Q_t(x(t))}, \tag{4.8}$$

which from (3.26) becomes

$$\psi(t) = t + \tfrac{1}{2} \cdot \frac{Q_t(x(t))}{f_0(x(t)) - t}. \tag{4.9}$$

Since $Q_t(x(t))$ has a second order zero at $t = \bar{v}$, it follows from Traub (Ref. 13, p. 25) that (4.9) is only linearly convergent. Because of the insight gained into the operation of the tangent method we propose to show this result for a slightly more general case.

Supposing that $Q_t(x(t))$ has an m'th order zero, where $m > 1$, at $t = \bar{v}$, we can write

$$Q_t(x(t)) = (t - \bar{v})^m g(t) \tag{4.10}$$

where it is assumed that $g(\bar{v}) \neq 0$. From (4.9) observe that $\psi(\bar{v}) = \bar{v}$. By direct differentiation of (4.8) and (4.10) and appropriate substitutions we find that

$$\psi^{(1)}(t) = 1 - \frac{\left(\frac{d}{dt}Q_t(x(t))\right)^2}{\left(\frac{d}{dt}Q_t(x(t))\right)^2} + \frac{(t - \bar{v})^m g(t)(t - \bar{v})^{m-2}[m(m-1)g(t) + \ldots]}{(t - \bar{v})^{2m-2}[mg(t) + \ldots]^2}, \tag{4.11}$$

and since $m = 2$ this expression simplifies to

$$\psi^{(1)}(\bar{v}) = \frac{m-1}{m} = \tfrac{1}{2}, \tag{4.12}$$

at $t = \bar{v}$. Consequently by Theorem 4.1, we conclude that (4.9) is linearly convergent with an asymptotic error constant of $\frac{1}{2}$.

A comparison of (4.11) and (4.15) gives some insight into the reason why the tangent method, discussed in the next section, is quadratically convergent.

4.5 The Tangent Method

The tangent method was first considered by Kowalik *et al.*,[7] following a brief remark by Morrison[6] where he deduced its form on the basis of geometrical arguments. In this case the iteration function is given by

$$\psi(t) = t + \frac{Q_t(x(t))}{f(x(t)) - t}. \tag{4.13}$$

and only differs from the Newton formula (4.9) by a factor of $\frac{1}{2}$. Lootsma[10] has observed that (4.13) is super-linearly convergent.

We treat the general case of $Q_t(x(t))$ having an m'th order zero at $t = \bar{v}$, where $m > 1$. Referring to equation (4.8), suppose that the second term is multiplied by m, so that the new iteration function is

$$\psi(t) = t - m\frac{Q_t(x(t))}{\frac{d}{dt}Q_t(x(t))}. \tag{4.14}$$

In this case it is easy to see $\psi(\bar{v}) = \bar{v}$, and by analogy with (4.11) it follows that

$$\psi^{(1)}(t) = 1 - m\frac{[Q_t^{(1)}(x(t))]^2}{[Q_t^{(1)}(x(t))]^2} + m\frac{(t - \bar{v})^m g(t)(t - \bar{v})^{m-2}[m(m-1)g(t) + \ldots]}{(t - \bar{v})^{2m-2}[mg(t) + \ldots]^2}. \tag{4.15}$$

Consequently,

$$\psi^{(1)}(\bar{v}) = 1 - m + m\cdot\frac{m-1}{m} = 0, \tag{4.16}$$

for every $m > 1$, and in addition

$$\psi^{(2)}(\bar{v}) = \frac{\left.\frac{d^3}{dt^3}Q_t(x(t))\right|_{t=\bar{v}}}{\left.\frac{d^2}{dt^2}Q_t(x(t))\right|_{t=\bar{v}}} \neq 0, \tag{4.17}$$

where $\left.\frac{d^2}{dt^2}Q_t(x(t))\right|_{t=\bar{v}}$ is given by (3.27), and

$$\left.\frac{d^3}{dt^3}Q_t(x(t))\right|_{t=\bar{v}} = -2\left.\frac{d^2}{dt^2}f_0(x(t))\right|_{t=\bar{v}}, \tag{4.18}$$

with the derivative on the right hand side being given in lemma 3.2.

Recalling, $Q_t(x(t))$ has a second order zero at $t = \bar{v}$, it follows from theorem 4.1 that (4.13) is quadratically convergent, and the asymptotic error coefficient is given by $\psi^{(2)}(\bar{v})/2$.

From the above analysis we are able to draw slightly stronger conclusions than Lootsma[10]; namely that the tangent method is quadratically convergent. In addition we can see that in the more general case where $Q_t(x(t))$ has an arbitrary m'th order zero at $t = \bar{v}$, the more general algorithm (4.14) is always quadratically convergent. The tangent method can be viewed as an application of the generalized Newton formula (4.14) whose discovery, together with its unique property, Traub[13] attributes to Schröder, who reported it about 1870.

4.4 The Interpolation Polynomial Method

Numerical experience[10] with the tangent method has shown that it tends to overshoot the minimum value $\bar{v}$ quite readily for some problems, which causes the algorithm to fail unless some *ad hoc* procedure such as reverting to the Morrison algorithm is used. This has led the author to formulate further super-linearly convergent methods which may have more stable convergence behaviour in a finite neighbourhood of the minimum point $\bar{v}$.

We now show that algorithms based on the quadratic Newton interpolation formula are super-linearly convergent. Only an outline, which is sufficient for our purposes, of the algorithms will be given. A detailed discussion, including a consideration of suitable starting procedures, will be the subject of a further report.

To facilitate the discussion the following notation is introduced. The function $S(t)$ is defined as $S(t) = \sqrt{Q_t(x(t))}$ for all real values of t. In addition supposing that $t = t_i$ we define $S_i = S(t_i)$ and $Q_i = Q_{ti}(x(t_i))$. The quadratic Newton interpolation polynomial for the function $S(t)$ at the points t_i, t_{i-1}, t_{i-2} with corresponding function values S_i, S_{i-1}, S_{i-2} is given by

$$P_S(t) = S_i + (t - t_i)S(t_i, t_{i-1}) + (t - t_i)(t - t_{i-1})S(t_i, t_{i-1}, t_{i-2}), \tag{4.19}$$

where the divided difference operators are defined by

$$S(t_i, t_{i-1}) = \frac{S_i - S_{i-1}}{t_i - t_{i-1}}, \text{ and} \tag{4.20}$$

$$S(t_i, t_{i-1}, t_{i-2}) = \frac{S(t_i, t_{i-1}) - S(t_{i-1}, t_{i-2})}{t_i - t_{i-2}}. \tag{4.21}$$

The polynomial $P_Q(t)$ is defined in a similar manner using the points t_i, t_{i-1}, t_{i-2} and the function values Q_i, Q_{i-1}, Q_{i-2}.

Algorithm A

Step 0. Select t_0, t_1, t_2 such that $t_0 < t_1 < t_2 < \bar{v}$, and set $i = 2$.

Step 1. Compute the solutions S_{i-j} given by

$$S_{i-j} = \min_{x \in R^n} S(t_{i-j}), \quad j = 0, 1, 2. \tag{4.22}$$

Step 2. Find t_M, where

$$t_M = \frac{t_i + t_{i-1}}{2} - \frac{S(t_i, t_{i-1})}{2S(t_i, t_{i-1}, t_{i-2})}. \tag{4.23}$$

Step 3. If $P_S(t_M) \geqslant 0$ set $t_{i+1} = t_M$.
If $P_S(t_M) < 0$ then set

$$t_{i+1} = t_M - \sqrt{t_M^2 - \frac{(S_i - S(t_i, t_{i-1})t_i + S(t_i, t_{i-1}, t_{i-2})t_i t_{i-1})}{S(t_i, t_{i-1}, t_{i-2})}} \tag{4.24}$$

Step 4. If $S_{i+1} = 0$ set $\bar{x} = \bar{x}(t_{i+1})$ and stop; otherwise increment parameter i and return to step 1.

Algorithm B

This is essentially the same as algorithm A excepting that function $Q_t(x(t))$ replaces $S(t)$ and polynomial $P_Q(t)$ replaces $P_S(t)$.

Of course the iteration function for each of these algorithms is no longer a simple expression. However it should be observed that in a sufficiently small neighbourhood of $\bar{v}$, the function $Q_t(x(t))$ can be written as

$$Q_t(x(t)) = \frac{1}{1+\bar{\beta}}(t - \bar{v})^2 + O[(t - \bar{v})^3]. \tag{4.25}$$

Consequently $Q(t_i, t_{i-1}, t_{i-2}) > 0$ which implies that t_M is the minimum of a quadratic polynomial. A similar argument applies for function $S(t)$. In addition the following lemma shows that in a sufficiently small neighbourhood of $\bar{v}$, the polynomial $P_S(t)$ (or $P_Q(t)$) has a real zero for $t < \bar{v}$, so that for this case the iteration function, which will be denoted by $\psi_S(t)(\psi_Q(t))$, is given by (4.24).

LEMMA 4.1. For a sufficiently small $\epsilon > 0$ suppose $S^{(3)}(t)$ [or $Q_t^{(3)}(x(t))$] is continuous and non-zero on the interval $(\bar{v} - \epsilon, \bar{v})$. Let the points $t_{i-2}, t_{i-1}, t_i \in (\bar{v} - \epsilon, \bar{v})$ satisfy the conditions $t_{i-2} < t_{i-1} < t_i < \bar{v}$. If $S^{(3)}(\bar{v}) \geqslant k > 0$, $(Q_{\bar{v}}^{(3)}(\bar{x}) \geqslant k > 0)$, then $P_S(t)$ $(P_Q(t))$ has a real root t_{i+1} such that $t_i < t_{i+1} < \bar{v}$, so that the sequence $t_i, i = 1, 2, \ldots$ is monotone increasing.

Proof. Since $S^{(3)}(\bar{v}) \geqslant k > 0$ it follows from continuity that there exists an $\epsilon > 0$ such that $S^{(3)}(t) > 0$ for $t \in (\bar{v} - \epsilon, \bar{v})$. We take $t_{i-2}, t_{i-1}, t_i \in (\bar{v} - \epsilon, \bar{v})$. The error equation for the interpolation polynomial $P_S(t)$ is given by Reference 15,

$$P_S(t) = S(t) - \frac{S^{(3)}(\xi(t))}{3!}\prod_{j=0}^{2}(t - t_j), \tag{4.26}$$

where $\xi(t)$ lies in the interval $(\bar{v} - \epsilon, \bar{v})$. Consequently

$$P_S(\bar{v}) = -\frac{S^{(3)}(\xi(\bar{v}))}{3!}\prod_{j=0}^{2}(\bar{v} - t_j), \tag{4.27}$$

and from the result that $S^{(3)}(t) > 0$, we have $P_S(\bar{v}) < 0$. In addition from property (*b*) in Section 3 we see that $P_S(t_i) > 0$. From the continuity of $P_S(t)$ it follows that it possesses a real zero t_{i+1} in the interval $t_i < t_{i+1} < \bar{v}$, and that the sequence $t_i, i = 1, 2, \ldots$ is monotone increasing.

THEOREM 4.2. Suppose $f_i(\cdot) \in C^2(R^n; R^1)$, $i = 0, 1, \ldots, m$, problem MP is a convex programming problem, and there exists a Kuhn-Tucker point $(\bar{x}, \bar{u})$ which satisfies the Jacobian uniqueness conditions. In addition, for Algorithms A and B suppose that $S^{(3)}(\bar{v}) \geqslant k > 0$, and $Q_t^{(3)}(x(t))\big|_{t=\bar{v}} \geqslant k > 0$ respectively. Then the iteration function $\psi_S(t)$ has an order $p \simeq 1{\cdot}84$ with an approximate asymptotic error coefficient

$$\left[\frac{S^{(3)}(\bar{v})}{3!\,S^{(1)}(\bar{v})}\right]^{0\cdot 42},$$

and the iteration function $\psi_Q(t)$ has an order $p \simeq 1{\cdot}26$, with an approximate asymptotic error coefficient

$$\left[\frac{1}{3}\frac{Q_t^{(3)}(x(t))\big|_{t=\bar{v}}}{Q_t^{(1)}(x(t))\big|_{t=\bar{v}}}\right]^{0\cdot 26}.$$

Proof. We restrict consideration to Algorithm A, as the proof for Algorithm B follows along analogous lines.

From lemma 4.1 it follows that in a sufficiently small neighbourhood of $\bar{v}$ the sequence of parameters $t_i, i = 1, 2, \ldots$ are determined by (4.24), which gives a root of the polynomial $P_S(t)$. In addition from theorem 3.3 and the definition of $S(t)$ we note that $S(t)$ has a zero of multiplicity one at $\bar{v}$. Traub (Ref. 13, Theorem 7.5) has shown that the order, p, of such an algorithm is given by the unique real positive root of

$$t^3 - \frac{1}{m}(t^2 + t + 1) = 0, \tag{4.28}$$

and the asymptotic error coefficient is

$$\left[\frac{m!\,S^{(3)}(\bar{r})}{3!\,S^{(m)}(\bar{r})}\right]^{(p-1)(m-1)}$$

where m is the multiplicity of the zero of $S(t)$ at $\bar{r}$. In this case $m = 1$, and consequently the order $p \simeq 1 \cdot 84$.

5. CONCLUDING REMARKS

In this report the convergence of a number of iterative procedures for method of centres algorithms has been investigated. The tangent method, which was previously known to be super-linearly convergent, has been shown to be quadratically convergent, and the known results for the Staha and Morrison methods were proved as an easy consequence of the present approach. In addition, the rate of convergence for two quadratic interpolation polynomial methods developed by the author was studied. It was shown that both methods are super-linearly convergent, with Algorithm A, in which the function $Q_t(x(t))$ is transformed into a function having a simple zero, being almost quadratically convergent. Limited numerical experience with Algorithm B, where the function $Q_t(x(t))$ is interpolated directly, has shown it to perform very favourably in comparison with the tangent method. It is proposed to extend these comparisons in the light of the above theoretical results.

The key to the main results obtained in this work has been the demonstration that the function $Q_t(x(t))$ has a zero of multiplicity two at the point $t = \bar{r}$. While $Q_t(x)$ has the special form given by (2.4), it appears that similar results concerning the multiplicity of the zero of $Q_t(x(t))$ can be obtained for a generalization of (2.4), provided, suitable differentiability assumptions are made.

Apart from the concrete results obtained above, the insight gained about the analytic properties of $Q_t(x(t))$ leads us to speculate on suitable alternative algorithms for finding the zeros of this function. Such methods as inverse interpolation and Halley's method,[13] as well as other methods of transforming $Q_t(x(t))$ into a function having a zero of multiplicity one are worthy of further consideration.

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